

→ This is nothing but the rotation around \hat{z} -axis with angle $\phi = \omega t$!

But, there's a weird thing.

- $\langle \vec{s} \rangle_{2\pi} = \langle \vec{s} \rangle_0$; It's ok.

$$|\alpha, \pm\rangle = U(\pm) [|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|] |\alpha\rangle \quad \parallel \phi = \omega t,$$

$$= e^{-\frac{i\phi}{2}} |\uparrow\rangle\langle\uparrow| |\alpha\rangle + e^{\frac{i\phi}{2}} |\downarrow\rangle\langle\downarrow| |\alpha\rangle$$


 $|\alpha, 2\pi\rangle = \underbrace{-}_{\text{---}} |\alpha, 0\rangle$. 

The state comes back with a minus sign!

on.
↳ precession period $\tau = \frac{2\pi}{\omega}$ for $\langle \vec{s} \rangle$.

but $\tau_{\text{stateket}} = \frac{4\pi}{\omega}$ for $|\alpha\rangle$.

(3) Generalization: SU(2) vs. SO(3)

• Pauli two-component formalism

with the "Pauli" spinor.

ket

$$|\uparrow\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_\uparrow, \quad |\downarrow\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_\downarrow$$

$$\langle \downarrow | \doteq (1, 0) \equiv \chi_\uparrow^\dagger, \quad \langle \uparrow | \doteq (0, 1) \equiv \chi_\downarrow^\dagger$$

a state

$$|\alpha\rangle \doteq \begin{pmatrix} \langle \uparrow | \alpha \rangle \\ \langle \downarrow | \alpha \rangle \end{pmatrix}, \quad \langle \alpha | \doteq (\langle \alpha | \uparrow \rangle, \langle \alpha | \downarrow \rangle)$$

\Rightarrow two-component "Pauli" spinor.

$$\underline{\underline{X}} = \begin{pmatrix} \langle \uparrow | \alpha \rangle \\ \langle \downarrow | \alpha \rangle \end{pmatrix} = \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix} = c_\uparrow X_\uparrow + c_\downarrow X_\downarrow$$

and

$$\underline{\underline{X}}^+ = (\langle \alpha | \uparrow \rangle, \langle \alpha | \downarrow \rangle) = (c_\uparrow^*, c_\downarrow^*)$$

- Pauli Matrices

def. $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow \tilde{\sigma}_k \doteq \frac{\hbar}{2} \sigma_k$$

cex. $\langle S_x \rangle = \langle \alpha | S_x | \alpha \rangle = \frac{\hbar}{2} \underline{\underline{X}}^+ \sigma_x \underline{\underline{X}}$ ← Try to verify this.

\Rightarrow properties :
$$\begin{cases} \sigma_i^2 = 1 \\ \{ \sigma_i, \sigma_j \} = 2 \delta_{ij} \\ [\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k \end{cases}$$

also, $\sigma_i^\dagger = \sigma_i$: Hermitian.

$$\begin{cases} \det[\sigma_i] = -1 : \text{"special"} \\ \text{Tr}[\sigma_i] = 0 : \text{traceless.} \end{cases}$$

Now, consider a vector $\vec{X} = (x, y, z)$ in the basis of CR

Pauli matrices :

$$\underline{\underline{X}} = x \sigma_1 + y \sigma_2 + z \sigma_3$$

$$= \begin{pmatrix} z & x - iy \\ xt + iy & -z \end{pmatrix}$$

: Hermitian, traceless.

length of the vector $|\vec{x}|^2 = x^2 + y^2 + z^2 = \det X$. 19

\rightarrow A rotation can be described by a unitary transformation,

$$X' = U X U^{-1}, \quad \parallel \quad \det U = 1$$

$X' = \vec{x}' \cdot \vec{\sigma}$

$\Rightarrow |\vec{x}'|^2 = |\vec{x}|^2 \Leftrightarrow \det X' = \det X$.

$\therefore U$ (a 2×2 matrix) is a rotation matrix,

mapping $SU(2)$ [U] onto $SO(3)$ [R].

special \hookrightarrow dimension of unitary "defining" representation, $\rightarrow 2 \times 2$ matrix
 $\det U = 1$ fundamental"

Since U is a 2×2 matrix, it can be written as

$$U = q_{\vec{b}_0} + i \vec{\sigma} \cdot \vec{q}_{\vec{b}} \quad \parallel \quad \vec{q}_{\vec{b}} = (q_1, q_2, q_3)$$

see HW 2.1

$$U U^\dagger = 1 \Rightarrow |q_{\vec{b}_0}|^2 + |\vec{q}_{\vec{b}}|^2 + i \vec{\sigma} \cdot (\vec{q}_{\vec{b}} q_{\vec{b}_0} - c.c.) + i \vec{\sigma} \cdot (\vec{q}_{\vec{b}} \times \vec{q}_{\vec{b}}^*) = 1$$

* use the identity $(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i \vec{\sigma} \cdot (\vec{a} \times \vec{b})$.

\Rightarrow $q_{\vec{b}_0}$ and $\vec{q}_{\vec{b}}$ are real. (to remove $\vec{\sigma}$ -dependence)
 chosen to be.

$$\cdot \quad q_{\vec{b}_0}^2 + |\vec{q}_{\vec{b}}|^2 = 1.$$

Choosing $q_{\vec{b}_0} = \cos \frac{1}{2}\Theta$, $\vec{q}_{\vec{b}} = -\hat{z} \sin \frac{1}{2}\Theta$,

$$U \times U^{-1} \rightarrow \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \\ z \end{pmatrix}$$

It rotates \vec{x} by Θ around z -axis.

$$\text{Check: } U = \cos \frac{\theta}{2} - i \hat{n}_3 \sin \frac{\theta}{2} = \exp \left[-i \frac{\theta}{2} \hat{n}_3 \right] = \exp \left[-i \frac{\tilde{\zeta}_2}{\hbar} \theta \right].$$

a general rotation by angle ϕ around \hat{n} -axis.

$$: q_{\vec{b}\circ} = \cos \frac{1}{2}\phi, \quad \vec{q}_{\vec{b}} = -\hat{n} \sin \frac{1}{2}\phi$$

$$\rightarrow U = \exp \left[-i \frac{1}{2} (\vec{n} \cdot \vec{\sigma}) \phi \right]. \quad \text{|| This is the case where } \vec{j} = \frac{\vec{n}}{2},$$

Verification

$$U = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left(\frac{\phi}{2} \right)^n (\hat{n} \cdot \vec{\sigma})^n \quad \text{by using } (\vec{a} \cdot \vec{b})(\vec{c} \cdot \vec{d}) = \vec{a} \cdot \vec{b} + i \vec{a} \cdot (\vec{c} \times \vec{d}) \\ \Rightarrow (\hat{n} \cdot \vec{\sigma})^{2n} = 1$$

$$= \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\phi}{2} \right)^{2k} \right] \cdot I - i(\hat{n} \cdot \vec{\sigma}) \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\phi}{2} \right)^{2k+1} \right] \\ = \cos \frac{\phi}{2} \cdot I - i(\hat{n} \cdot \vec{\sigma}) \sin \frac{\phi}{2}.$$

* U has the period of 4π !. Does it sound reasonable?

Yes. $SU(2)$ covers $SO(3)$ twice!

"Cayley-Klein" parameters

$$\text{In another general form, } U(a, b) = \begin{pmatrix} a & b \\ -b & a^* \end{pmatrix}$$

$$\text{with } |a|^2 + |b|^2 = 1.$$

$$\Rightarrow U^+(a, b) \times U(a, b) = X' \quad \parallel U(-a, b) \\ U^+(-a, -b) \times U(-a, -b) = X'$$

U and $-U$ generates the same R .

$$[2\pi] + [2\pi] \longrightarrow [2\pi].$$

: A state ket rotated by U has 4π -periodicity!

$$= U(\hat{n}, \phi) |\alpha\rangle \quad \text{in } SU(2).$$

(4) Eigenvalues and Eigenstates of \vec{J} and J^2 .

- Commutation Relations and Ladder Operators

Lie Algebra : $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$

Casimir operator

Every thing starts from this relation.

$$\Rightarrow [J^2, J_k] = 0. \quad \parallel J^2 = J_x^2 + J_y^2 + J_z^2$$

: There are simultaneous eigenvectors of J^2 and J_k .

$$\Rightarrow J^2 |a, b\rangle = a |a, b\rangle$$

$$J_z |a, b\rangle = b |a, b\rangle$$

|| NOTE: There's a typo
in S&N. indeed.

def. Ladder operators ; let's see how these work.

$$J_{\pm} \equiv J_x \pm i J_y$$

$$\text{Commutation relations : } [J_+, J_-] = 2i\hbar J_z$$

$$[J_z, J_{\pm}] = \pm i\hbar J_{\pm}$$

Why "ladder"?

$$[J^2, J_{\pm}] = 0.$$

$$\Rightarrow J_z (J_{\pm} |a, b\rangle) = ([J_z, J_{\pm}] + J_{\pm} J_z) |a, b\rangle$$

$$= (b \pm \hbar) (J_{\pm} |a, b\rangle)$$

: It raises or lowers the eigenvalue b .

BUT it doesn't change "a" since $[J^2, J_{\pm}] = 0$.

$$\Rightarrow J^2 (J_{\pm} |a, b\rangle) = a (J_{\pm} |a, b\rangle)$$

Therefore, we may write it as

$$J_{\pm} |a, b\rangle = c_{\pm} |a, b \pm \hbar\rangle$$

c-number

- Eigenvalues of J^2 and J_z .

Can we apply J_{\pm} again and again, indefinitely? No.

$$\text{Consider } J^2 - J_z^2 = \frac{1}{2} (J_+ J_- + J_- J_+) \quad \dots \text{ (1)}$$

$$= \frac{1}{2} (J_-^\dagger J_- + J_+^\dagger J_+) \quad \dots \text{ (2)}$$

$$\Rightarrow \langle a, b | J^2 - J_z^2 | a, b \rangle = \frac{1}{2} [\langle -|- \rangle + \langle +|+ \rangle] \quad \dots \text{ (3)}$$

$$\stackrel{(1)}{\geq} 0 \quad \parallel \quad \begin{aligned} |- \rangle &= J_-(a, b) \\ |+ \rangle &= J_+(a, b) \end{aligned}$$

$$\Rightarrow \underbrace{a \geq b^2}_{\text{b has upper and lower bounds given by a.}} \quad \dots \text{ (4)}$$

$$\text{Thus, } J_+ |a, b_{\max}\rangle = 0 \Rightarrow J_- J_+ |a, b_{\max}\rangle = 0 \quad \dots \text{ (5)}$$

$$\Rightarrow \underbrace{(J^2 - J_z^2 - \hbar J_z)}_{\text{from (1)}} |a, b_{\max}\rangle = 0 \quad \dots \text{ (6)}$$

$$\text{proof. } J_- J_+ = J_x^2 + J_y^2 - \hbar J_z = J^2 - J_z^2 - \hbar J_z$$

$$\Rightarrow a - b_{\max}^2 - b_{\max} \hbar = 0$$

$$\text{or } a = b_{\max} (b_{\max} + \hbar) \quad \dots \text{ (7)}$$

$$\text{Similarly, } J_- |a, b_{\min}\rangle = 0 \Rightarrow J_+ J_- |a, b_{\min}\rangle = 0 \quad \dots \text{ (8)}$$

$$\Rightarrow (J^2 - J_z^2 + \hbar J_z) |a, b_{\min}\rangle = 0 \quad \dots \text{ (9)}$$

$$\Rightarrow a = b_{\min} (b_{\min} - \hbar) \quad \dots \text{ (10)}$$

$$\rightarrow \text{ (7) - (10) : } (b_{\max}^2 - b_{\min}^2) + \hbar (b_{\max} + b_{\min}) = 0.$$

$$\therefore \boxed{b_{\min} = -b_{\max}} \quad \dots \text{ (11)}$$

Since we can reach b_{\max} by applying J_+ to $|b_{\min}\rangle$

a finite number of times,

$$\underline{b_{\max} = b_{\min} + nh} .$$

Define

$$\underline{j = \frac{n}{2}} = \frac{1}{2}, 1, \frac{3}{2}, \dots$$

\Downarrow
 $n : \text{integer, } > 0$

$$b_{\max} = \frac{nh}{2}$$

$$b_{\min} = -\frac{nh}{2}$$

↓

$$\underline{m = \frac{1}{2} j(j+1) \text{ and } b = nh}$$

let,

The allowed $m = -j, -j+1, \dots, j-1, j$

$$\therefore \begin{cases} J^2 |j, m\rangle = j(j+1)h^2 |j, m\rangle & \text{if } j = \frac{1}{2}, 1, \frac{3}{2}, \dots \\ J_z |j, m\rangle = nh |j, m\rangle & \text{a half integer!} \end{cases}$$

: This is a direct outcome of the Lie Algebra;
We did not use anything else.

(b) Matrix elements of \vec{J} and $D(R)$.

- J^2, J_z, J_{\pm}

$$\text{obviously, } \langle j', m' | J^2 | j, m \rangle = j(j+1)h^2 \delta_{jj'} \delta_{mm'}$$

$$\langle j', m' | J_z | j, m \rangle = nh \delta_{jj'} \delta_{mm'}$$

For J_+ , we know $J_+ |j, m\rangle = \langle_{jm}^{(+)} |j, m+1\rangle$.

$$\begin{aligned} \Rightarrow \langle j, m | J_+^{\dagger} J_+ | j, m \rangle &= \langle j, m | (J^2 - J_z^2 - nh J_z) | j, m \rangle \\ &= h^2 [j(j+1) - m^2 - m] \end{aligned}$$

$$\therefore \left| C_{jm}^{(+)} \right|^2 = h^2 [j(j+1) - m^2 - m] = \underline{h^2 (j-m)(j+m+1)}$$

Choosing $C_{jm}^{(\pm)}$ to be real and positive,

$$J_{\pm}|j,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} \hat{h} |j, m \pm 1\rangle$$

(You can check this for J_- similarly.)

$$\Rightarrow \langle j', m' | J_{\pm} | j, m \rangle = \underbrace{\sqrt{(j \mp m)(j \pm m + 1)}}_{\text{---}} \cdot \underbrace{\hat{h}}_{\text{---}} \cdot \underbrace{S_{j,j}}_{\text{---}} \underbrace{S_{m',m \pm 1}}_{\text{---}}$$

• Representations of the Rotation Operator

$$\mathcal{D}_{m'm}^{(j)}(R) = \langle j, m' | \exp \left[-\frac{i}{\hbar} (\vec{J} \cdot \hat{n}) \phi \right] | j, m \rangle$$

(Wigner function): a matrix element of $\mathcal{D}(R)$
on cl-matrix:

NOTE: it's diagonal in $|j\rangle$. $\parallel \vec{J}|j\rangle \propto |j\rangle$

\rightarrow a block-diagonal matrix

$$\begin{bmatrix} \square & & & & \\ & \square & & & \\ & & \ddots & & \\ & & & \square & \\ & & & & \ddots \end{bmatrix}$$

$(m = -j \dots j)$

\rightarrow $-(2j+1) \times (2j+1)$

The rotation matrices characterized by definite j : 

form a "group".

\parallel NOTE: 2 is the dimension of the
on $SU(2)$ "defining, fundamental" one.

- Identity: $\phi = 0$.

- Inverse: $\phi \rightarrow -\phi$

- Composition:

$$\sum_{m''} \mathcal{D}_{m''m'}^{(j)}(R_1) \mathcal{D}_{m'm}^{(j)}(R_2) = \mathcal{D}_{m'm}^{(j)}(R_1 R_2)$$